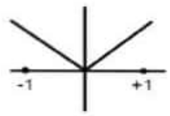
Continuity & Differentiability

Some Fundamental Definitions

A function $f(x)$ is defined in the interval I, then it is said to be continuous at a point $x = a$ if $\lim_{x \to a} f(x) = f(a)$

A function $f(x)$ is said to be differentiable at $x = a$ if $\lim_{h \to 0} \frac{f(x+h) - f(a)}{h} = f'(a)$ *exists* $\quad a \in \mathrm{I}$

Ex : Consider a function $f(x)$ is defined in the interval [-1,1] by $f(x)$ = $|x| = \begin{cases} -x & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$

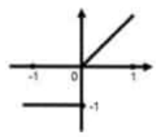


It is continuous at $x = 0$
But not differentiable at $x = 0$

Note : If a function $f(x)$ is differentiable then it is continuous, but converse need not be true.
Geometrically :

(1) If $f(x)$ is Continuous at $x = a$ means, $f(x)$ has no breaks or jumps at the point $x = a$

Ex : $\quad f(x) = \begin{cases} -1 & -1 \le x \le 0 \\ x & 0 < x \le 1 \end{cases}$



Is discontinuous at x=0

(2) If $f(x)$ is differentiable at $x = a$ means, the graph of $f(x)$ has a unique tangent at the point or graph is smooth at $x = a$

1. **Give the definitions of Continuity & Differentiability:**

Solution: A function f (x) is said to be continuous at x = a, if corresponding to an arbitrary positive number ε, however small, their exists another positive number δ such that.

$|f(x) - f(a)| < \varepsilon$, where $|x - a| < \delta$

It is clear from the above definition that a function f (x) is continuous at a point 'a'.

If (i) it exists at x = a

(ii) $\underset{x \to a}{Lt} f(x) = f(a)$

i.e, limiting value of the function at x = a is to the value of the function at x = a

Differentiability:

A function f (x) is said to be differentiable in the interval (a, b), if it is differentiable at every point in the interval.

In Case [a,b] the function should posses derivatives at every point and at the end points a & b i.e., Rf^1 (a) and Lf^1 (a) exists.

2. **State Rolle's Theorem with Geometric Interpretation.**

Statement: Let f (x) be a function is defined on [a,b] & it satisfies the following Conditions.

(i) f (x) is continuous in [a,b]

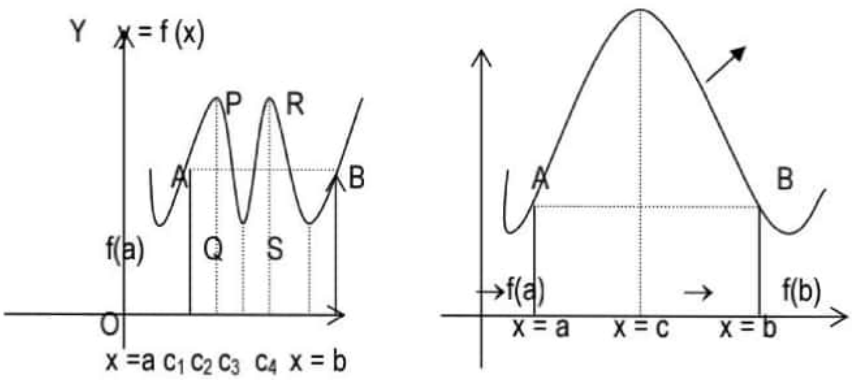
(ii) f (x) is differentiable in (a,b)

(iii) f (a) = f (b)

Then there exists at least a point $C \in (a,b)$, Here $a < b$ such that $f^1(c) = 0$

Proof:

Geometrical Interpretation of Rolle's Theorem:



Let us consider the graph of the function y = f (x) in xy – plane. A (a,.f(a)) and B (*b*, *f*(*b*)) be the two points in the curve f (x) and a, b are the corresponding end points of A & B respectively. Now, explained the conditions of Rolle's theorem as follows.

(i) f (x) is continuous function in [a,b], Because from figure without breaks or jumps in between A & B on y = f (x).

(ii) f (x) is a differentiable in (a,b), that means let us joining the points A & B, we get a line AB.

$\therefore$ Slope of the line AB = 0 then $\exists$ a point C at P and also the tangent at P (or Q or R or S) is Parallel to x –axis.

$\therefore$ Slope of the tangent at P (or Q or R or S) to be Zero even the curve y = f (x) decreases or increases, i.e., f (x) is Constant.

$f^1(x) = 0$

$\therefore f^1(c) = 0$

(iii) The Slope of the line AB is equal to Zero, i.e., the line AB is parallel to x – axis.

$\therefore f(a) = f(b)$

3. Verify Rolle's Theorem for the function f (x) = $x^2 - 4x + 8$ in the internal [1,3]

Solution: We know that every Poly nominal is continuous and differentiable for all points and hence f (x) is continuous and differentiable in the internal [1,3].

Also $f(1) = 1 - 4 + 8 = 5$, $f(3) = 3^2 - 43 + 8 = 5$

Hence $f(1) = f(3)$

Thus f (x) satisfies all the conditions of the Rolle's Theorem. Now $f^1(x) = 2x - 4$ and $f^1(x) = 0$ $\Rightarrow 2x - 4 = 0$ or $x = 2$. Clearly $1 < 2 < 3$. Hence there exists 2t (1,3) such that $f^1(2) = 0$. This shows that Rolle's Theorem holds good for the given function f (x) in the given interval.

4. Verify Rolle's Theorem for the function f (x) = x (x + 3) $e^{-x/2}$ in the interval [-3, 0]

Solution: Differentiating the given function W.r.t 'x' we get

$$f^1(x) = (x^2 + 3x)\left(-\frac{1}{2}\right)e^{-x/2} + (2x+3)e^{-x/2}$$

$$= -\frac{1}{2}(x^2 - x - 6)e^{-x/2}$$

$\therefore f^1(x)$ exists (i.e finite) for all x and hence continuous for all x.

Also $f(-3) = 0$, $f(0) = 0$ so that $f(-3) = f(0)$ so that $f(-3) = f(0)$. Thus f (x) satisfies all the conditions of the Rolle's Theorem.

Now, $f^1(x) = 0$

$$\Rightarrow -\frac{1}{2}(x^2 - x - 6)e^{-x/2} = 0$$

Solving this equation we get x = 3 or x = -2

Clearly $-3 < -2 < 0$. Hence there exists $-2 \in (-3,0)$ such that $f^1(-2) = 0$

This proves that Rolle's Theorem is true for the given function.

5. Verify the Rolle's Theorem for the function Sin x in [-π, π]

Solution: Let f (x) = Sin x

Clearly Sinx is continuous for all x.

Also $f^1(x)$ = Cos x exists for all x in (-π, π) and f (-π) = Sin (-π) = 0; f (π) = Sin (π) = 0 so that f (-π) = f (π)

Thus f (x) satisfies all the conditions of the Rolle's Theorem .

Now $f^1(x) = 0 \Rightarrow \cos x = 0$ so that

$$X = \pm \frac{\pi}{2}$$

Both these values lie in (-π,π). These exists $C = \pm \frac{\pi}{2}$

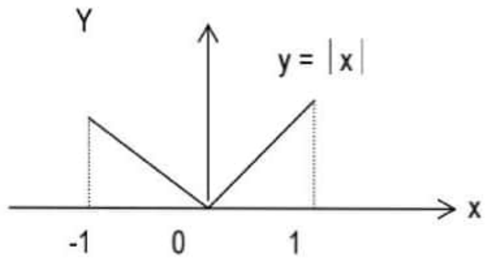
Such that $f^1(c) = 0$

Hence Rolle's theorem is vertified.

6. Discuss the applicability of Rolle's Theorem for the function f (x) = lxl in [-1,1].

Solution: Now $f(x) = |x| = \begin{cases} x \text{ for } 0 \le x \le 1 \\ -x \text{ for } -1 \le x \le 0 \end{cases}$

f (x) being a linear function is continuous for all x in [-1, 1]. f(x) is differentiable for all x in (-1,1) except at x = 0. Therefore Rolle's Theorem does not hold good for the function f (x) in [-1,1]. Graph of this function is shown in figure. From which we observe that we cannot draw a tangent to the curve at any point in (-1,1) parallel to the x – axis.



Exercise:

7. **Verify Rolle's Theorem for the following functions in the given intervals.**

   a) $x^2 - 6x + 8$ in [2,4]

   b) $(x - a)^3 (x - b)^3$ in [a,b]

   c) $\log \left\{ \frac{x^2 + ab}{(a+b)x} \right\}$ in [a,b]

8. **Find whether Rolle's Theorem is applicable to the following functions. Justify your answer.**

   a) $f(x) = |x - 1|$ in [0,2]

   b) $f(x) = \tan x$ in $[0, \pi]$.

9. **State & prove Lagrange's (1st) Mean Value Theorem with Geometric meaning.**

Statement: Let f (x) be a function of x such that

(i) If is continuous in [a,b]

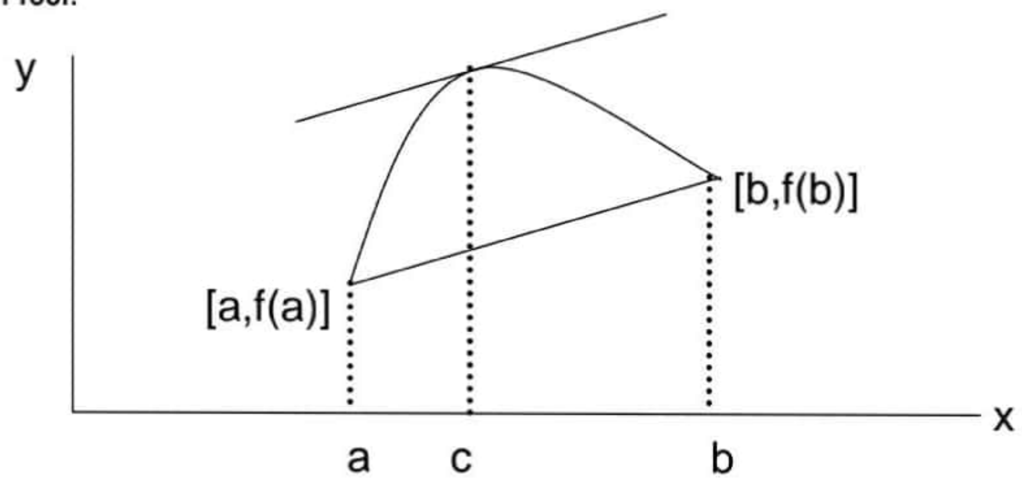
(ii) If is differentiable in (a,b)

Then there exists atleast a point (or value) $C \in (a,b)$ such that.

$$f^1(c) = \frac{f(b) - f(a)}{b - a}$$

i.e., $f(b) = f(a) + (b - a) f^1(c)$

Proof:



Define a function g (x) so that g (x) = f (x) – Ax ---------- (1)

Where A is a Constant to be determined.

So that g (a) = g (b)

Now, g (a) = f (a) – Aa

G (b) = f (b) – Ab

$\therefore$ g (a) = g (b) $\Rightarrow$ f (a) – Aa = f (b) – Ab.

i.e., $A = \frac{f(b) - f(a)}{b - a}$ ---------------- (2)

Now, g (x) is continuous in [a,b] as rhs of (1) is continuous in [a,b]

G(x) is differentiable in (a,b) as r.h.s of (1) is differentiable in (a,b).

Further g (a) = g (b), because of the choice oif A.

Thus g (x) satisfies the conditions of the Rolle's Theorem.

$\therefore$ These exists a value x = c sothat a < c < b at which $g^1(c) = 0$

$\therefore$ Differentiate (1) W.r.t 'x' we get

$g^1(x) = f^1(x) - A$

$\therefore g^1(c) = f^1(c) - A$ ($\because$ x =c)

$\Rightarrow f^1(c) - A = 0$ ($\because g^1(c) = 0$)

$\therefore f^1(c) = A$ -------------- (3)

From (2) and (3) we get

$f^1(c) = \frac{f(b) - f(a)}{b - a}$ (or) $f(b) = f(a) + (b - a) f^1(c)$ For a < c < b

Corollary: Put b – a = h

i.e., b = a + h and c = a + θ h

Where 0 < θ < 1

Substituting in $f(b) = f(a) + (b - a) f^1(c)$

$\therefore f(a + h) = f(a) + h f(a + \theta h)$, where $0 < \theta < 1$.

Geometrical Interpretation:-

Since y = f (x) is continuous in [a,b], it has a graph as shown in the figure below,
At x = a, y = f (a)
At x = b, y = f (b)

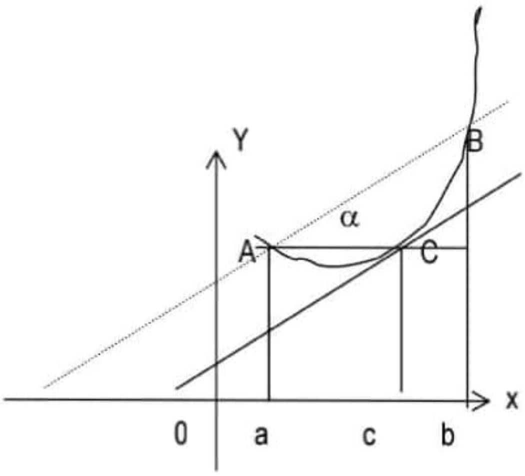


Figure (i)

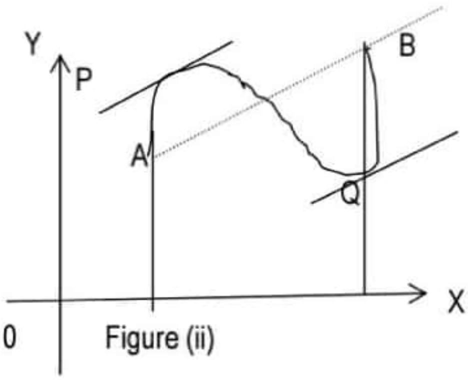


Figure (ii)

Slope of the line joining the points A (a,f(a)) and B ((b,f (b))

Is $\frac{f(b)-f(b)}{b-a}$ $\quad$ ($\therefore$ Slope = m = tan θ)

= tan α

Where α is the angle mode by the line AB with x – axis

= Slope of the tangent at x = c

= f1 (c), where a < c < b

Geometrically, it means that there exists at least are value of x = c, where a < c < b at which the tangent will be parallel to the line joining the end points at x = a & x = b.

Note: These can be more than are value at which the tangents are parallel to the line joining points A & B (from Fig (ii)).

10. **Verify Lagrange's Mean value theorem for f(x) = (x – 1) (x – 2) (x – 3) in [0,4].**

Solution: Clearly given function is continuous in [0,4] and differentiable in (0,4), because f (x) is in polynomial.

$f(x) = (x-1)(x-2)(x-3)$

$f(x) = x^3 - 6x^2 + 11x - 6$

and $f(0) = 0^3 - 6(0)^2 + 11(0) - 6 = -6$

$f(4) = 4^3 - 6(4)^2 + 11(4) - 6 = 6$

Differentiate f (x) W.r.t x, we get

$F^1(x) = 3x^2 - 6x + 11$

Let x = c, $f^1(c) = 3c^2 - 6c + 11$

By Lagrange's Mean value theorem, we have

$$f^1(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(0)}{(4-0)}$$

$$= \frac{6-(-6)}{4} = 3$$

$\therefore 3c^2 - 6c + 11 = 3$

$\Rightarrow 3c^2 - 6c + 8 = 0$

Solving this equation, we get

$$C = 2 \pm \frac{2}{\sqrt{3}} \in (0,4)$$

Hence the function is verified.

11. **Verify the Lagrange's Mean value theorem for f (x) = logx in [1,e].**

Solution: Now Logx is continuous for all x > 0 and hence [1,e].

Also $f^1(x) = \frac{1}{x}$ which exists for all x in (1,e)

Hence f (x) is differentiable in (1,e)

$\therefore$ by Lagrange's Mean Value theorem, we get

$$\frac{Loge - Log1}{e-1} = \frac{1}{c} \Rightarrow \frac{1}{e-1} = \frac{1}{c}$$

$\Rightarrow C = e - 1$

$\Rightarrow 1 < e - < 2 < e$

Since $e \in (2,3)$

$\therefore$ So that c = e – 1 lies between 1 & e

Hence the Theorem.

12. Find θ for f (x) = $Lx^2 + mx + n$ by Lagrange's Mean Value theorem.

Solution: f (x) = $Lx^2 + mx + n$

$\therefore f^1(x) = 2Lx + m$

We have $f(a+h) = f(a) + hf^1(a + \theta h)$

Or $f(a+h) - f(a) = hf^1(a + \theta h)$

i.e., $\{\ell(a+h)^2 + m(a+h) + n\} - \{\ell a^2 + ma + n\} = h\{2\ell(a + \theta h) + m\}$

Comparing the Co-efficient of ℓh^2, we get

$1 = 2\theta$ $\quad\quad \therefore \theta = \frac{1}{2} \in (0,1)$

Exercise:

13. Verify the Lagrange's Mean Value theorem for f (x) = Sin^2x in $\left[0, \frac{\pi}{2}\right]$

14. Prove that, $\frac{b-a}{1+b^2}$ **<** $\tan^{-1} b - \tan^{-1} a$ **<** $\frac{b-a}{1+a^2}$ **if 0 < a < b and reduce that**

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

15. Show that $\frac{2}{\pi} < \frac{Sinx}{x} < 1$ **in** $\left(0, \frac{\pi}{2}\right)$

16. Prove that $\frac{b-a}{\sqrt{1-a^2}} < Sin^{-1}b - Sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$ Where a < b. Hence reduce

$$\frac{\pi}{6} - \frac{1}{2\sqrt{3}} < Sin^{-1}\frac{1}{4} < \frac{\pi}{6} - \frac{1}{\sqrt{15}}.$$

17. State & prove Cauchy's Mean Value Theorem with Geometric meaning.

Proof: The ratio of the increments of two functions called Cauchy's Theorem.
Statement: Let g (x) and f (x) be two functions of x such that,

(i) Both f (x) and g (x) are continuous in [a,b]

(ii) Both f (x) and g (x) are differentiable in (a,b)

(iii) $g^1(x) \neq 0$ for any x ε (a,b)

These three exists at least are value x = c ε (a,b) at which

$$\frac{f^1(c)}{g^1(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof: Define a function,

$\phi(x) = f(x) - A.\ g(x)$ ------------------ (1)

So that $\phi(a) = \phi(b)$ and A is a Constant to be determined.

Now, $\phi(a) = f(a) - A\ g(a)$

$\phi(b) = f(b) - A.\ g(b)$

$\therefore f(a) - A\ g(a) = f(b) - A.\ g(b)$

$\Rightarrow A = \frac{f(b)-f(b)}{g(b)-g(a)}$ -------------------- (2)

Now, ϕ is continuous in [a,b] as r.h.s of (1) is continuous in [a,b] and $\phi(x)$ is differentiable in (a,b) as r.h.s of (1) is differentiable in [a,b].

Also $\phi(a) = \phi(b)$

Hence all the conditions of Rolle's Theorem are satisfied then there exists a value $x = c \in (a,b)$ such that $\phi 1(c) = 0$.

Now, Differentiating (1) W.r.t x, we get

$\phi^1(x) = f^1(x) - A.g^1(x)$

at $x = c \in (a,b)$

$\therefore \phi^1(c) = f^1(c) - A g^1(c)$

$0 = f^1(c) - A g^1(c)$ $(\because g^1(x) \neq 0)$

$\Rightarrow A = \dfrac{f^1(c)}{g^1(c)}$ ---------------- (3)

Substituting (3) in (2), we get

$$\frac{f^1(c)}{g^1(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}, \text{ where } a < c < b$$

Hence the proof.

18. Verify Cauchy's Mean Value Theorem for the function f (x) = $x^2 + 3$, g (x) = $x^3 + 1$ in [1,3]

Solution: Here f (x) = $x^2 + 3$, g (x) = $x^3 + 1$

Both f (x) and g (x) are Polynomial in x. Hence they are continuous and differentiable for all x and in particular in [1,3]

Now, $f^1(x) = 2x$, $g^1(x) = 3x^2$

Also $g^1(x) \neq 0$ for all $x \in (1,3)$

Hence f (x) and g (x) satisfy all the conditions of the cauchy's mean value theorem. Therefore

$$\frac{f(3) - f(1)}{g(3) - g(1)} = \frac{f^1(c)}{g^1(c)}, \text{ for some } c : 1 < c < 3$$

i.e., $\dfrac{12 - 4}{28 - 2} = \dfrac{26}{3c^2}$

i.e., $\dfrac{2}{13} = \dfrac{1}{3c} \Rightarrow C = \dfrac{13}{6} = 2\dfrac{1}{6}$

Clearly C = $2\frac{1}{6}$ lies between 1 and 3.

Hence Cauchy's theorem holds good for the given function.

19. Verify Cauchy's Mean Value Theorem for the functions f (x) Sin x, g (x) = Cos x in $\left[0, \frac{\pi}{2}\right]$

Solution: Here f (x) = Sin x, g (x) = Cos x so that

$f^1(x) = \text{Cos } x, g^1(x) = -\text{Sin} x$

Clearly both f (x) and g (x) are continuous in $\left[0, \frac{\pi}{2}\right]$, and differentiable in $\left(0, \pi/2\right)$

Also $g^1(x) = -\text{Sin } x \neq 0$ for all $x \in \left(0, \pi/2\right)$

∴ From cauchy's mean value theorem we obtain

$$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{g\left(\frac{\pi}{2}\right) - g(0)} = \frac{f^1(c)}{g^1(c)} \text{ for some } C : 0 < C < \frac{\pi}{2}$$

i.e., $\frac{1-0}{0-1} = \frac{Cosc}{-Sinc}$ i.e., -1 = - Cot c (or) Cot c = 1

∴ C = $\frac{\pi}{4}$, clearly C = $\frac{\pi}{4}$ lies between 0 and $\frac{\pi}{2}$

Thus Cauchy's Theorem is verified.

Exercises:

20. Find C by Cauchy's Mean Value Theorem for

a) f (x) = e^x, g (x) = e^{-x} in [0,1]

b) f (x) = x^2, g (x) = x in [2,3]

21. Verify Cauchy's Mean Value theorem for

a) f (x) = $\tan^{-1} x$, g (x) = x in $\left[\frac{1}{\sqrt{3}}, 1\right]$

b) f(x) = log x, g (x) = $\frac{1}{x}$ in [1,e]

Generalized Mean Value Theorem:

22. State Taylor's Theorem and hence obtain Maclaurin's expansion (series)

Statement: If f (x) and its first (n – 1) derivatives are continuous in [a,b] and its n^{th} derivative exists in (a,b) then

$$f(b) = f(a) + (b-a) f^1(a) + \frac{(b-a)^2}{2!} f^{11}(a) + \text{---------} + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(c)$$

Where a < c < b

Remainder in Taylor's Theorem:

We have

$$f(x) = f(a) + (x-a) f^1(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \text{---------} + \frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n}{\angle n} f^n[a + (x-a)\theta\}$$

$$f(x) = S_n(x) + R_n(x)$$

Where $R_n(x) = \frac{(x-a)^n}{n!} f^n[a + (x-a)\theta]$ is called the Largrange's form of the Remainder.

Where a = 0, $R_n(x) = \frac{x^n}{n!} f^n(\theta x)$, $0 < \theta < 1$

Taylor's and Maclaurin's Series:

We have $f(x) = S_n(x) + R_n(x)$

$$\therefore \lim_{n\to\infty}[f(x) - S_n(x)] = \lim_{n\to\infty} R_n(x)$$

If $\lim_{n\to\infty} R_n(x) = 0 then f(x) \lim_{n\to\infty} S_n(x)$

Thus $\lim_{n\to\infty} S_n(x)$ converges and its sum is f (x).

This implies that f (x) can be expressed as an infinite series.

i.e., $f(x) = f(a) + (x-a) f^1(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \text{----------}$ to ∞

This is called Taylor's Series.

Putting a = 0, in the above series, we get

$$F(x) = f(0) + x f^{1}(0) + \frac{(x)^2}{2!} f^{11}(0) + \text{---------} \text{ to } \infty$$

This is called Maclaurin's Series. This can also denoted as

$$Y = y(0) + x y_1(0) + \frac{(x)^2}{2!} y_2(0) + \text{--------} + \frac{(x)^n}{n!} y_n(0) \text{----------} \text{ to } \infty$$

Where $y = f(x)$, $y_1 = f_1(x)$, -------------- $y_n = f_n(x)$

23. By using Taylor's Theorem expand the function e^x in ascending powers of (x – 1)

Solution: The Taylor's Theorem for the function f (x) is ascending powers of (x – a) is

$$f(x) = f(a) + (x-a) f^{1}(a) + \frac{(x-a)^2}{\angle 2} f^{11}(a) + \text{------------} \quad (1)$$

Here $f(x) = e^x$ and a = 1

$f^{1}(x) = e^x \Rightarrow f^{1}(a) = e$

$f^{11}(x) = e^x \Rightarrow f^{11}(a) = e$

$\therefore$ (1) becomes

$$e^x = e + (x-1) e + \frac{(x-1)^2}{2} e + \text{-------------}$$

$$= e \left\{ 1 + (x-1) + \frac{(x-1)^2}{2} + \text{--------} \right\}$$

24. By using Taylor's Theorem expand log sinx in ascending powers of (x – 3)

Solution: f (x) = Log Sin x, a = 3 and f (3) = log sin3

$$\text{Now } f^{1}(x) = \frac{Cosx}{Sinx} = \text{Cotx}, \; f^{1}(3) = \text{Cot3}$$

$f^{11}(x) = -\text{Cosec}^2 x$, $f^{11}(3) = -\text{Cosec}^2 3$

$f^{111}(x) = -2\text{Cosecx}(-\text{Cosecx Cotx}) = 2\text{Cosec}^3 x \text{ Cotx}$

$\therefore f^{111}(3) = 2\text{Cosec}^3 3 \text{ Cot3}$

$$\therefore f(x) = f(a) + (x-a) f^{1}(a) + \frac{(x-a)^2}{2!} f^{11}(a) + \frac{(x-a)^3}{3!} f^{111}(a) + \text{------------}$$

$$\therefore \text{Log Sinx} = f(3) + (x-3) f^{1}(3) + \frac{(x-3)^2}{2!} f^{11}(3) + \frac{(x-3)^3}{3!} f^{111}(3) + \text{-----------}$$

$$= \text{logsin3} + (x-3)\text{ Cot3} + \frac{(x-3)^2}{2!} (-\text{Cosec}^2 3) + \frac{(x-3)^3}{3!} 2\text{ Cosec}^3 3\text{Cot3} + \text{---}$$

Exercise:

25. **Expand Sinx is ascending powers of $\left(x - \frac{\pi}{2}\right)$**

26. **Express $\tan^{-1} x$ in powers of $(x - 1)$ up to the term containing $(x - 1)^3$**

27. **Apply Taylor's Theorem to prove**

$$e^{x+h} = e^x \left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + ----------\right]$$

Problems on Maclaurin's Expansion:

28. **Expand the log (1 + x) as a power series by using Maclaurin's theorem.**

Solution: Here f (x) = log (1 + x), Hence f (0) = log 1 = 0

We know that

$$f^n(x) = \frac{d^n}{dx^n}\{\log(1+x)\} = \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{1}{1+x}\right\}$$

$$= \frac{(-1)^{n-1}(n-1)!}{(1+n)^n}, \qquad n = 1,2,---------$$

Hence $f^n(0) = (-1)^{n-1}(n-1)!$

$f^1(0) = 1$, $f^{11}(0) = -1$, $f^{111}(0) = 3!$, $f^{1v}(0) = -3!$

Substituting these values in

$$f(x) = f(0) + x f^1(0) + \frac{x^2}{2!} f^{11}(0) + -------- + \frac{x^n}{n!} f^n(0) + ------------$$

$$\therefore \log(1+x) = 0 + x \cdot 1 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!} 2! + \frac{x^4}{4!} - 3! + ---------------$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ---------$$

This series is called Logarithmic Series.

29. Expand $\tan^{-1} x$ by using Macluarin's Theorem up to the term containing x^5

Solution: let $y = \tan^{-1} x$, Hence $y(0) = 0$

We find that $y_1 = \frac{1}{1+x^2}$ which gives $y_1(0) = 1$

Further $y_1(1 + x^2) = 1$, Differentiating we get

$Y_1 . 2x + (1 + x^2) y_2 = 0$ (or) $(1 + x^2) y_2 + 2xy_1 = 0$

Hence $y_2(0) = 0$

Taking n^{th} derivative an both sides by using Leibniz's Theorem, we get

$(1 + x^2) y_{n+2} + n . 2xy_{n+1} + \frac{n(n-1)}{1.2} . 2. y_n + 2xy_{n-1} + n.2.y_n = 0$

i.e., $(1 + x^2) y_{n+2} + 2(n+1) x y_{n+1} + n(n+1) y_n = 0$

Substituting $x = 0$, we get, $y_{n+2}(0) = -n(n+1) y_n(0)$

For $n = 1$, we get $y_3(0) = -2y_1(0) = -2$

For $n = 2$, we get $y_4(0) = -2.3.y_2(0) = 0$

For $n = 3$, we get $y_5(0) = -3.4.y_3(0) = 24$

Using the formula

$Y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \text{---------}$

We get $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \text{-------------}$

Exercise:

30. Using Maclaurin's Theorem prove the following:

a) $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \text{--------}$

b) $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \text{--------------}$

c) $e^x \cos x = 1 + x - \frac{x^3}{3} + \text{---------------}$

d) Expand $e^{ax} \cos bx$ by Maclaurin's Theorem as far as the term containing x^3

Exercise : Verify Rolle's Theorem for

(i) $f(x) = e^x(\sin x - \cos x)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$,

(ii) $f(x) = x(x-2)e^{x/2}$ in $[0,2]$

(iii) $f(x) = \frac{\sin 2x}{e^{2x}}$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Exercise : Verify the Lagrange's Mean Value Theorem for

(i) $f(x) = x(x-1)(x-2)$ in $\left[0, \frac{1}{2}\right]$

(ii) $f(x) = Tan^{-1}x$ in $[0,1]$

Exercise : Verify the Cauchy's Mean Value Theorem for

(i) $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in $\left[\frac{1}{4}, 1\right]$

(ii) $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$ in $[a,b]$

(iii) $f(x) = Sin\ x$ and $g(x) = Cos\ x$ in $[a,b]$

DIFFERENTIAL CALCULUS-II

Give different types of Indeterminate Forms.

If f (x) and g (x) be two functions such that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exists, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ then

$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ Which do not have any definite value, such an expression is called indeterminate form. The other indeterminate forms are $\frac{\infty}{\infty}, 0\times\infty, \infty-\infty$, 0^0, ∞^0 and 1^{∞}

1. State & prove L' Hospital's Theorem (rule) for Indeterminate Forms.

L'Hospital rule is applicable when the given expression is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Statement: Let f (x) and g (x) be two functions such that

(1) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

(2) f^1 (a) and g^1 (a) exist and $g^1(a) \neq 0$

Then $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f^1(x)}{\lim_{x\to a} g^1(x)}$

Proof: Now $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\left[\frac{1/g\,x}{1/f(x)}\right]$, which takes the indeterminate form $\frac{0}{0}$. Hence applying the L' Hospitals theorem, we get

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{-g^1(x)\Big/[g(x)]^2}{-f^1(x)\Big/[f(x)]^2} = \lim_{x\to a}\frac{g^1(x)}{f^1(x)}\left[\frac{f(x)}{g(x)}\right]^2$$

$$= \left[\lim_{x\to a}\frac{g^1(x)}{f^1(x)}\right]\left[\lim_{x\to a}\frac{f(x)}{g(x)}\right]^2$$

If $\lim_{x\to a}\frac{f(x)}{g(x)} \neq 0 and \neq \infty$ then

$$1 = \left[\lim_{x\to a}\frac{g^1(x)}{f^1(x)}\right]\left[\lim_{x\to a}\frac{f(x)}{g(x)}\right]$$

i.e $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f^1(x)}{g^1(x)}$

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ = 0 or ∞ the above theorem still holds good.

2. Evaluate $\lim_{x\to a}\frac{Sinx}{x} = \frac{0}{0}$ **form**

Solution: Apply L'Hospital rule, we get

$$\lim_{x\to a}\frac{Cosx}{1} = \frac{Cos\theta}{1} = \frac{1}{1} = 1$$

$$\therefore \lim_{x\to a}\frac{Sinx}{x} = 1$$

3. Evaluate $\lim_{x\to a}\frac{\log Sinx}{Cotx}$

Solution: $\lim_{x\to a}\frac{\log Sinx}{Cotx} = \frac{\log Sin0}{Cot0} = \frac{\log 0}{\infty} = \frac{-\infty}{\infty}$ form

Apply L' Hospital rule

$$= \lim_{x\to a}\frac{-Cosce^2 x}{-2Co\sec xCo\sec xCotx}$$

$$= \lim_{x\to a} \frac{1}{2Cotx} = 0$$

$$\therefore \lim_{x\to a} \frac{\log Sinx}{Cotx} = 0$$

Exercise: 1

Evaluate

a) $\lim_{x\to 0} \frac{\tan x}{x}$

b) $\lim_{x\to 0} (1+x)^{1/x}$

c) $\lim_{x\to \infty} \frac{a^x - 1}{x}$

d) $\lim_{x\to 0} \frac{x^n - a^n}{x-a}$

4. Explain $\infty \cdot \infty$ and $0 \times \infty$ Forms:

Solution: Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ in this case

$\lim_{x\to a} f(x)$ - g(x) = $0 \times \infty$, reduce this to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form

Let $\lim_{x\to a} [f(x).g(x)] = \lim_{x\to a} \left\{ \frac{f(x)}{1/g(x)} \right\} = \frac{0}{0}$ form

Or $\lim_{x\to a} [f(x).g(x)] = \lim_{x\to a} \left\{ \frac{g(x)}{1/f(x)} \right\} = \frac{\infty}{\infty}$ form

L' Hospitals rule can be applied in either case to get the limit.

Suppose $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ in this case $\lim_{x\to a} [f(x).g(x)] = \infty - \infty$ form, reduce this $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form and then apply L'Hospitals rule to get the limit

5. **Evaluate** $\lim_{x\to 0}\left\{\frac{1}{x}-\frac{\log(1+x)}{x^2}\right\}$

Solution: Given $\lim_{x\to 0}\left\{\frac{1}{x}-\frac{\log(1+x)}{x^2}\right\}$ = ∞ - ∞ form

$\therefore$ Required limit = $\lim_{x\to 0}\left\{\frac{x-\log(1+x}{x^2}\right\}=\frac{0}{0}$ form

Apply L'Hospital rule.

$$= \lim_{x\to 0}\frac{1-\frac{1}{1+x}}{2x}$$

$$= \lim_{x\to 0}\frac{\frac{x}{1+x}}{2x}=\lim_{x\to 0}\frac{1}{2(1+x)}=\frac{1}{2}$$

6. **Evaluate** $\lim_{x\to 0}\left\{\frac{1}{x}-Cotx\right\}$

Solution: Given limit is ∞ - ∞ form at x = 0. Hence we have

Required limit = $\lim_{x\to 0}\left\{\frac{1}{x}-\frac{Cosx}{Sinx}\right\}$

$$= \lim_{x\to 0}\left(\frac{Sinx-xCosx}{xSinx}\right)=\left(\frac{0}{0}\right) \text{ form}$$

Apply L' Hospital's rule

$$= \lim_{x\to 0}\frac{Cosx-Cosx+xSinx}{xCosx+Sinx}$$

$$= \lim_{x\to 0}\frac{xSinx}{xCosx+Sinx}=\left(\frac{0}{0}\right) \text{ form}$$

Apply L' Hospitals rule

$$= \lim_{x\to 0} \frac{xCosx + Sinx}{Cosx - xSinx + Cosx}$$

$$= \frac{0}{2-0} = \frac{0}{2} = 0$$

7. Evaluate $\lim_{x\to 0}$ tan x log x

Solution: Given limit is $(0 \times -\infty)$ form at x = 0

$\therefore$ Required limit = $\lim_{x\to 0} \frac{\log x}{Cotx} = \left[\frac{-\infty}{\infty}\right]$ form

Apply L' Hospitals rule

$$= \lim_{x\to 0} \frac{1/x}{-Co\sec^2 x}$$

$$= \lim_{x\to 0} \frac{-Sin^2 x}{x} \left(\frac{0}{0}\right) \text{ form}$$

Apply L Hospitals rule

$$= \lim_{x\to 0} \frac{-2SinxCosx}{1} = 0$$

8. Evaluate $\lim_{x\to 1} Sec\left(\frac{\pi x}{2}\right).\log x$

Solution: Given limit is $(\infty \times 0)$ form at x = 1

$\therefore$ Required limit = $\lim_{x\to 1} \frac{\log x}{Cos\frac{\pi x}{2}} \frac{0}{0}$ form

Apply L' Hospitals rule

$$= \lim_{x\to 1} \frac{1/x}{-Sin\frac{\pi x}{2}.\frac{\pi}{2}} = -\frac{2}{\pi}$$

Exercise: 2

Evaluate

a) $\underset{x\to 1}{Lim}\left(\frac{x}{x-1}-\frac{1}{\log x}\right)$

b) $\underset{x\to 0}{Lim}\left(\frac{a}{x}-Cot\left(\frac{x}{a}\right)\right)$

c) $\underset{x\to \frac{\pi}{2}}{Lim}\left(Secx-\frac{1}{1-Sinx}\right)$

d) $\underset{x\to \infty}{Lim}\left(a^{1/x}-1\right)x$

9. Explain Indeterminate Forms 0^0, 1^∞, ∞^0, 0^∞

Solution: At x = a, $[f(x)]^{g(x)}$ takes the indeterminate form

(i) 0^0 if $\underset{x\to a}{Lim}$ f(x) = 0 and $\underset{x\to a}{Lim}$ g (x) = 0

(ii) 1^∞ if $\underset{x\to a}{Lim}$ f(x) = 1 and $\underset{x\to a}{Lim}$ g (x) = ∞

(iii) ∞^0 if $\underset{x\to a}{Lim}$ f(x) = ∞ and $\underset{x\to a}{Lim}$ g(x) = 0 and

(iv) 0^∞ if $\underset{x\to a}{Lim}$ f(x) = 0 and $\underset{x\to a}{Lim}$ f (x) = ∞

In all these cases the following method is adopted to evaluate $\underset{x\to a}{Lim}\ [f(x)]^{g(x)}$

Let L = $\underset{x\to a}{Lim}\ [f(x)]^{g(x)}$ so that

Log L = $\underset{x\to a}{Lim}$ [g(x) log f (x)] = $0\times\infty$

Reducing this to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and applying L' Hospitals rule, we get Log L = a Or L = e^a

10. Evaluate $\lim_{x\to 0} x^{\sin x}$

Solution: let L = $\lim_{x\to 0} x^{\sin x} \Rightarrow 0^0$ form at x = 0

Hence Log L = $\lim_{x\to 0} \sin x \log x \Rightarrow 0 \times \infty$ form

$$\therefore LogL = \lim_{x\to 0} \frac{\log x}{1/\sin x} = \lim_{x\to 0} \frac{\log x}{Cosec x}\left(\frac{\infty}{\infty}\right) \text{ form}$$

Apply L' Hospital rule,

$$= \lim_{x\to 0} \frac{1/x}{-Cosec x Cot x} = \lim_{x\to 0} \frac{-\sin x.\tan x}{x}\left(0/0\right) \text{ form}$$

Apply L' Hospitals rule we get

$$= \lim_{x\to 0} \frac{\sin x Sec^2 x - Cos x \tan x}{1} = -\infty$$

$$\therefore LogL = -\infty \Rightarrow L = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$\therefore$ L = 0

11. **Evaluate** $\lim_{x\to 1} (x)^{1/1-x}$

Solution: let L = $\lim_{x\to 1} (x)^{1/1-x}$ is 1^{∞} form

$$\therefore LogL = \lim_{x\to 1}\left(\frac{1}{1-x}\log x\right) \equiv \left(\frac{0}{0}\right) \text{ form}$$

Apply L' Hospitals rule

$$= \lim_{x\to 1} \frac{1/x}{-1} = \lim_{x\to 1} \frac{1}{-x} = -1$$

$\therefore$ Log L = -1

$\Rightarrow$ L = $e^{-1} = \frac{1}{e}$

12. Evaluate $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2}$

Solution: let L = $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2} \equiv 1^{\infty}$ form

$$\therefore LogL = \lim_{x \to 0} \left\{ \frac{1}{x^2} \log\left(\frac{\tan x}{x} \right) \right\} \equiv (\infty \times 0) \text{ form}$$

$$\therefore LogL = \lim_{x \to 0} \left\{ \frac{\log\left(\tan x/x\right)}{x^2} \right\} \equiv \left(\frac{0}{0} \right) \text{ form}$$

Apply L' Hospitals rule

$$= \lim_{x \to 0} \left\{ \frac{1}{\frac{\tan x}{x}} \frac{xSec^2 x - \tan x}{x^2} \Big/ 2x \right\}$$

$$LogL = \frac{1}{2} \lim_{x \to 0} \left[\frac{xSec^2 x - \tan x}{x^3} \right] = \left(\frac{0}{0} \right) \text{ form}$$

Apply L' Hospital rule, we get

$$= \frac{1}{2} \lim_{x \to 0} \frac{Sec^2 x + 2xSec^2 x \tan x - Sec^2 x}{3x^2}$$

$$= \frac{1}{3} \lim_{x \to 0} (Sec^2 x) \left(\frac{\tan x}{x} \right)$$

Log L = $\frac{1}{3}$

$$\therefore L = e^{1/3}$$

Exercise: 3

Evaluate the following limits.

a) $\lim_{x\to 0}(Secx)^{Cotx}$

b) $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

c) $\lim_{x\to 0}\frac{(1+x)^{1/x}-e}{x}$

d) $\lim_{x\to 0}(Cosax)^{\infty/x^2}$

$(i)\ \lim_{x\to 0}\frac{e^x-e^{-x}-2\log(1+x)}{x\sin x}$

$(ii)\ \lim_{x\to 0}\frac{\log(1+x^3)}{\sin^3 x}$

$(iii)\ \lim_{x\to 0}\frac{1+\sin x-\cos x+\log(1-x)}{x\tan^2 x}$

$(iv)\ \lim_{x\to\frac{\pi}{2}}\frac{\log\sin x}{(x-\frac{\pi}{2})^2}$

$(v)\ \lim_{x\to 0}\frac{\cosh x+\log(1-x)-1+x}{x^2}$

$(vi)\ \lim_{x\to\frac{\pi}{2}}\frac{\sin x\sin^{-1}x}{x^2}$

$(vii)\ \lim_{x\to 0}\frac{e^{2x}-(1+x)^2}{x\log(1+x)}$

Evaluate the following limits.

$(i)\ \lim_{x\to a}\left(2-\frac{x}{a}\right)\cot(x-a)$

$(ii)\ \lim_{x\to 0}(\operatorname{cosec} x-\cot x)$

$(iii)\ \lim_{x\to\frac{\pi}{2}}\left[x\tan x-\frac{\pi}{2}\sec x\right]$

$(iv)\ \lim_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right)$

$(v)\ \lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{x\tan x}\right]$

$(vi)\ \lim_{x\to\frac{\pi}{2}}[2x\tan x-\pi\sec x]$

$(i)\ \lim_{x\to 0}(\cos ax)^{\frac{b}{x^2}}$

$(ii)\ \lim_{x\to 0}\left(\frac{1+\cos x}{2}\right)^{\frac{1}{x^2}}$

$(iii)\quad \lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ $\qquad (iv)\quad \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$(v)\quad \lim_{x\to 0}(\sin x)^{\tan x}$ $\qquad (iv)\quad \lim_{x\to 0}\left(1+\sin x\right)^{\cot x}$

$(vii)\quad \lim_{x\to 0}(\cos x)^{\operatorname{cosec}^2 x}$ $\qquad (viii)\quad \lim_{x\to \frac{\pi}{4}}\left(\tan x\right)^{\tan 2x}$

$(ix)\quad \lim_{x\to \infty}\left(\frac{ax+1}{ax-1}\right)^{x}$ $\qquad (x)\quad \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

Evaluate the following limits.

$(i)\quad \lim_{x\to 0}(\cos ax)^{\frac{b}{x^2}}$ $\qquad (ii)\quad \lim_{x\to 0}\left(\frac{1+\cos x}{2}\right)^{\frac{1}{x^2}}$

$(iii)\quad \lim_{x\to 1}(1-x^2)^{\frac{1}{\log(1-x)}}$ $\qquad (iv)\quad \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$(v)\quad \lim_{x\to 0}(\sin x)^{\tan x}$ $\qquad (iv)\quad \lim_{x\to 0}\left(1+\sin x\right)^{\cot x}$

$(vii)\quad \lim_{x\to 0}(\cos x)^{\operatorname{cosec}^2 x}$ $\qquad (viii)\quad \lim_{x\to \frac{\pi}{4}}\left(\tan x\right)^{\tan 2x}$

$(ix)\quad \lim_{x\to \infty}\left(\frac{ax+1}{ax-1}\right)^{x}$ $\qquad (x)\quad \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$